

FROM SIMPLE-MINDED COLLECTIONS TO SILTING OBJECTS VIA KOSZUL DUALITY

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ABSTRACT. Given an elementary simple-minded collection in the derived category of a non-positive dg algebra with finite-dimensional total cohomology, we construct a silting object via Koszul duality.

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1. INTRODUCTION

Projective modules, simple modules and the interaction between them are crucial in the representation theory of finite-dimensional algebras. In the context of triangulated category, silting objects and simple-minded collections are generalisations of projective modules and simple modules, respectively. Silting objects are ‘generators’ of co-t-structures ([1, 4, 17, 12]), and simple-minded collections are ‘generators’ of t-structures ([2, 13]).

In [18] (and [19, 13]), Rickard provided a method to construct a silting object of the bounded homotopy category $K^b(\text{proj } \Lambda)$ from a given simple-minded collection in the bounded derived category $\mathcal{D}^b(\text{mod } \Lambda)$, where Λ is a finite-dimensional algebra. In this paper, we provide an alternative approach to Rickard’s construction. Our approach works more generally for non-positive dg algebras with finite-dimensional total cohomology. For a simple-minded collection in the finite-dimensional derived category $\mathcal{D}_{fd}(A)$ of a non-positive dg algebra A with finite-dimensional total cohomology, we construct, by using the triangulated version of Koszul duality ([8, 14, 15]), a silting object of the perfect derived category $\text{per}(A)$. More precisely,

Theorem 1.1. *Let A be a non-positive dg algebra over a field k with finite-dimensional total cohomology. For an elementary simple-minded collection $\{X_1, \dots, X_r\}$ of $\mathcal{D}_{fd}(A)$, there exists a unique (up to isomorphism) silting object $M = M_1 \oplus \dots \oplus M_r$ of $\text{per}(A)$ such that for $1 \leq i, j \leq r$ and $p \in \mathbb{Z}$*

$$\text{Hom}_{\mathcal{D}_{fd}(A)}(M_i, \Sigma^p X_j) = \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, we obtain the following theorem, generalising [13, Theorems 6.1 and 7.12].

Theorem 1.2. *Let A be a non-positive dg algebra over an algebraically closed field with finite-dimensional total cohomology. There are one-to-one correspondences which commute with mutations and which preserve partial orders between*

- (1) *equivalence classes of silting objects in $\text{per}(A)$,*
- (2) *isomorphism classes of simple-minded collections in $\mathcal{D}_{fd}(A)$,*
- (3) *bounded t -structures of $\mathcal{D}_{fd}(A)$ with length heart,*
- (4) *bounded co- t -structures of $\text{per}(A)$.*

A ‘dual’ of this result, namely, the same statement with A replaced by a homologically smooth non-positive dg algebra with finite-dimensional cohomology in each degree was obtained in [12].

The paper is structured as follows. In Section 2, we recall the basics on silting objects and non-positive dg algebras. In Section 3, we recall the basics on A_∞ -algebras and A_∞ -modules. In Section 4, we recall the definition of simple-minded collections and study strictly unital minimal positive A_∞ -algebras, which are closely related to simple-minded collections. In Section 5 we provide the construction of a silting object from a given simple-minded collection.

Throughout the paper, let k be a field and let $D = \text{Hom}_k(?, k)$ be the k -dual. Without further remark, modules will be right modules and all categories are k -categories.

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2. SILTING OBJECTS AND NON-POSITIVE DG ALGEBRAS

In this section we recall the definitions and standard facts on silting objects and non-positive dg algebras.

2.1. Silting objects. Let \mathcal{C} be a triangulated category with suspension functor Σ . For a set \mathcal{S} of objects of \mathcal{C} , let $\text{add}(\mathcal{S}) = \text{add}_{\mathcal{C}}(\mathcal{S})$ be the smallest full subcategory of \mathcal{C} containing \mathcal{S} and closed under taking direct summands and finite direct sums, and let $\text{thick}(\mathcal{S}) = \text{thick}_{\mathcal{C}}(\mathcal{S})$ denote the smallest thick subcategory of \mathcal{C} containing \mathcal{S} .

An object M of \mathcal{C} is a *silting object* of \mathcal{C} if

- $\text{Hom}_{\mathcal{C}}(M, \Sigma^p M) = 0$ for any $p > 0$,
- $\mathcal{C} = \text{thick}(M)$.

Two silting objects M and M' of \mathcal{C} are said to be *equivalent* if $\text{add}(M) = \text{add}(M')$.

2.2. Non-positive dg algebras. Let A be a dg k -algebra. Denote by $\mathcal{D}(A)$ the derived category of (right) dg A -modules (see [8, 10]), which is a triangulated category with suspension functor Σ the shift functor. Let $\text{per}(A) = \text{thick}(A_A)$, the thick subcategory of $\mathcal{D}(A)$ generated by A_A , the free dg A -module of rank 1, and let $\mathcal{D}_{fd}(A)$ be the full subcategory of $\mathcal{D}(A)$ consisting of dg A -modules whose total cohomology is finite-dimensional. If A is a finite-dimensional k -algebra, then a dg A -module is exactly a complex of A -modules. So $\mathcal{D}(A) = \mathcal{D}(\text{Mod } A)$ and we have canonical triangle equivalences $K^b(\text{proj } A) \rightarrow \text{per}(A)$ and $\mathcal{D}^b(\text{mod } A) \rightarrow \mathcal{D}_{fd}(A)$. For two dg A -modules M and N , let $\mathcal{H}om_A(M, N)$ denote the complex whose degree p component

consists of those A -linear maps from M to N which are homogeneous of degree p , and whose differential takes f to $d_N \circ f - (-1)^{|f|} f \circ d_M$, where f is homogeneous of degree $|f|$.

We say that A is *non-positive* if $A^p = 0$ vanishes for all $p > 0$. A triangulated category is said to be *algebraic* if it is triangle equivalent to the stable category of a Frobenius category.

Lemma 2.1 ([13, Lemma 4.1]). (a) *Let A be a non-positive dg k -algebra. The free dg A -module of rank 1 is a silting object of $\text{per}(A)$.*

(b) *Let \mathcal{C} be an idempotent complete algebraic triangulated category and let $M \in \mathcal{C}$ be a silting object. Then there is a non-positive dg k -algebra A together with a triangle equivalence $\text{per}(A) \xrightarrow{\sim} \mathcal{C}$ which takes A to M .*

2.3. Cohomologically finite-dimensional non-positive dg algebras. Let A be a non-positive dg k -algebra whose total cohomology is finite-dimensional over k . Then A_A is a silting object of $\text{per}(A)$. Moreover, $\text{per}(A) \subseteq \mathcal{D}_{fd}(A)$ and $\text{thick}(D(AA)) \subseteq \mathcal{D}_{fd}(A)$. There is a triangle functor $\nu : \mathcal{D}(A) \rightarrow \mathcal{D}(A)$ (called the *Nakayama functor*) which restricts to a triangle equivalence $\nu : \text{per}(A) \rightarrow \text{thick}(D(AA))$. We have the Auslander–Reiten formula

$$D \text{Hom}(M, N) \cong \text{Hom}(N, \nu(M))$$

for $M \in \text{per}(A)$ and $N \in \mathcal{D}(A)$. See [8, Section 10].

Let M be a silting object of $\text{per}(A)$. We may assume that M is cofibrant ([10, Proposition 3.1], [8, Theorem 3.1]) and form the dg endomorphism algebra $\mathcal{E}nd_A(M) := \mathcal{H}om_A(M, M)$. Then by [8, Lemma 6.1], we have a derived equivalence $\mathcal{D}(\mathcal{E}nd_A(M)) \rightarrow \mathcal{D}(A)$ taking $\mathcal{E}nd_A(M)$ to M . Since $H^p \mathcal{E}nd_A(M) = \text{Hom}_{\mathcal{D}(A)}(M, \Sigma^p M)$, it follows that $\mathcal{E}nd_A(M)$ has finite-dimensional total cohomology and $H^p \mathcal{E}nd_A(M) = 0$ for all $p > 0$. The subcomplex $\sigma^{\leq 0} \mathcal{E}nd_A(M)$, where $\sigma^{\leq 0}$ is the standard truncation in degree 0, is a dg subalgebra of $\mathcal{E}nd_A(M)$. In particular, it is a non-positive dg algebra with finite-dimensional total cohomology. Moreover, the canonical embedding $\sigma^{\leq 0} \mathcal{E}nd_A(M) \rightarrow \mathcal{E}nd_A(M)$ is a quasi-isomorphism of dg algebras, inducing a derived equivalence

$$\mathcal{D}(\sigma^{\leq 0} \mathcal{E}nd_A(M)) \rightarrow \mathcal{D}(A)$$

taking $\sigma^{\leq 0} \mathcal{E}nd_A(M)$ to M . We call $\sigma^{\leq 0} \mathcal{E}nd_A(M)$ the *truncated dg endomorphism algebra* of M .

3. A_∞ -ALGEBRAS AND A_∞ -MODULES

In this section we recall the definition and basic properties of A_∞ -algebras and A_∞ -modules. We follow [14] and also refer to [9] and [15].

Let R be a separable semi-simple k -algebra. An A_∞ -algebra A over R is a graded R -bimodule endowed with a family of homogeneous R -bilinear maps $m_n : A^{\otimes_R n} \rightarrow A$ ($n \geq 1$) of degree $2 - n$, called the *multiplications* of A , which satisfies the following identities

$$\sum_{i+j+l=n} (-1)^{ij+l} m_{i+1+l}(\text{id}^{\otimes i} \otimes m_j \otimes \text{id}^{\otimes l}) = 0, n \geq 1. \quad (3.1)$$

Here $i, l \geq 0$ and $j \geq 1$. (We take \mathbf{C} as the category of R -bimodules in [14, Définition 1.2.1.1].) We are mainly interested in the case when R is a finite direct product of copies of k , which we use ‘as if it were non-commutative’, namely, we do not require that the left and right graded R -module structures on A coincide (compare *e.g.* [16, Section 2.1]). Let A be an A_∞ -algebra over R . A is said to be *strictly unital* if there is a R -bilinear map $\eta : R \rightarrow A$ (called the *unit* of A) which is homogeneous of degree 0 such that $m_n(\text{id} \otimes \cdots \otimes \text{id} \otimes \eta \otimes \text{id} \otimes \cdots \otimes \text{id}) = 0$ for $i \neq 2$ and $m_2(\text{id} \otimes \eta) = m_2(\eta \otimes \text{id}) = \text{id}$. Note that the identity (3.1) for $n = 1$ is $m_1^2 = 0$, thus A is a complex of R -bimodules with differential m_1 . A is said to be *minimal* if $m_1 = 0$. In this case, A is a graded algebra over R with m_2 as multiplication.

Let A and B be two strictly unital A_∞ -algebras over R . A *strictly unital A_∞ -morphism* $f : A \rightarrow B$ of strictly unital A_∞ -algebras is a family of homogeneous R -bilinear maps $f_n : A^{\otimes_R n} \rightarrow B$ ($n \geq 1$) of degree $1 - n$, such that $f_1 \eta_A = \eta_B$, $f_n(\text{id} \otimes \cdots \otimes \text{id} \otimes \eta_A \otimes \text{id} \otimes \cdots \otimes \text{id}) = 0$ for all $n \geq 2$, and that

$$\sum_{i+j+l=n} (-1)^{ij+l} f_{i+1+l}(\text{id}^{\otimes i} \otimes m_j \otimes \text{id}^{\otimes l}) = \sum_{\substack{1 \leq p \leq n \\ i_1 + \dots + i_p = n}} (-1)^\omega m_p(f_{i_1} \otimes \cdots \otimes f_{i_p}), n \geq 1. \quad (3.2)$$

Here $j \geq 1; i, l \geq 0$ and $\omega = \sum_{2 \leq u \leq p} (1 - i_u) \sum_{1 \leq v \leq u} i_v$. It follows that f_1 is a chain map with respect to the differentials m_1 . If f_1 is a quasi-isomorphism of complexes, we say that f is an *A_∞ -quasi-isomorphism*. If $f_n = 0$ for $n \geq 2$, then the above identities amounts to saying that $f_1 : A \rightarrow B$ commutes with all multiplications b_n . In this case, we say that f is *strict* and identify f with f_1 .

Let A be a strictly unital A_∞ -algebra over R . A is said to be *augmented* if there is a strict A_∞ -morphism $\varepsilon : A \rightarrow R$ of strictly unital A_∞ -algebras, which is called the *augmentation* of A . Here we view R as a strictly unital A_∞ -algebra over R with $m_2 = \text{id}$, $m_n = 0$ for $n \neq 2$ and $\eta = \text{id}$. Let A and B be two augmented A_∞ -algebras over R . An *A_∞ -morphism* $f : A \rightarrow B$ of augmented A_∞ -algebras is a strictly unital A_∞ -morphism of strictly unital A_∞ -algebras such that $\varepsilon_B f_1 = \varepsilon_A$.

A dg algebra A over R is a dg k -algebra together with a homomorphism $\eta : R \rightarrow A$ of dg k -algebras. It can be considered as a strictly unital A_∞ -algebra over R with m_1 being the differential, m_2 being the multiplication and $m_n = 0$ for $n \geq 3$.

Theorem 3.1 ([14, Proposition 7.5.0.2] and [14, Lemme 2.3.4.3]). *Let A be a strictly unital A_∞ -algebra over R . Then there is a dg algebra A' (called a dg model of A) over R with a strictly unital A_∞ -quasi-isomorphism $A \rightarrow A'$. If A is augmented over R , then A' can be taken augmented over R (in this case A' is called the enveloping dg algebra of A) and the A_∞ -quasi-isomorphism above is an A_∞ -quasi-isomorphism of augmented A_∞ -algebras over R .*

Let A be a strictly unital A_∞ -algebra over R . A (right) A_∞ -module over A is a graded right R -module M endowed with a family of homogeneous R -linear maps $m_n^M : M \otimes_R A^{\otimes_R n-1} \rightarrow M$ ($n \geq 1$) of degree $2 - n$ such that (some of the superscripts M on m are omitted)

$$\sum_{i+j+l=n} (-1)^{ij+l} m_{i+1+l}(\text{id}^{\otimes i} \otimes m_j \otimes \text{id}^{\otimes l}) = 0. \quad (3.3)$$

Here $i, l \geq 0$ and $j \geq 1$. The graded R -module M equipped with m_1^M becomes a complex. An A_∞ -module M is said to be *minimal* if $m_1^M = 0$. It is said to be *strictly unital* if $m_n^M(\text{id}_M \otimes \text{id} \otimes \cdots \otimes \text{id} \otimes \eta \otimes \text{id} \otimes \cdots \otimes \text{id}) = 0$ for all $n \geq 3$, and $m_2^M(\text{id}_M \otimes \eta) = \text{id}_M$. If M' is a graded R -submodule of M such that m_n^M restricts to M' for all $n \geq 1$, then M' together with the restriction of m_n^M is called a *submodule* of M . A together with its multiplications is an A_∞ -module over A . An element e of A is called a *strict idempotent* if $e \in \text{im}(\eta)$, $m_2(e \otimes e) = e$ and for all $n \neq 2$ we have $m_n(a_1 \otimes \cdots \otimes a_n) = 0$ if one of a_1, \dots, a_n is e . If e is a strict idempotent of A , then $eA = \{ea := m_2(e \otimes a) | a \in A\}$ is an A_∞ -submodule of A , because

$$m_n(ea_1 \otimes a_2 \otimes \cdots \otimes a_n) = em_n(a_1 \otimes a_2 \otimes \cdots \otimes a_n).$$

Let M and M' be two strictly unital A_∞ -modules over A . An *strictly unital A_∞ -morphism* $f : M \rightarrow M'$ is a family of homogeneous R -linear maps $f_n : M \otimes_R A^{\otimes n-1} \rightarrow M'$ ($n \geq 1$) of degree $1 - n$ such that $f_n(\text{id}_M \otimes \text{id} \otimes \cdots \otimes \text{id} \otimes \eta \otimes \text{id} \otimes \cdots \otimes \text{id}) = 0$ for all $n \geq 2$ and that the following identity holds for all $n \geq 1$

$$\sum_{i+j+l=n} (-1)^{ij+l} f_{i+1+l}(\text{id}^{\otimes i} \otimes m_j \otimes \text{id}^{\otimes l}) = \sum_{s+t=n} m_{1+t}(f_s \otimes \text{id}^{\otimes t}). \quad (3.4)$$

Here $i, l, t \geq 0$ and $j, s \geq 1$. In particular, f_1 is a chain map of complexes. f is an *A_∞ -quasi-isomorphism* if f_1 induces identities on all cohomologies. f is *strict* if $f_n = 0$ for all $n \geq 2$. We will identify f with f_1 in this case.

Proposition 3.2. ([14, Proposition 3.3.1.7]) *Let A be a strictly unital A_∞ -algebra over R and M be a strictly unital A_∞ -module over A . Then there is a strictly unital minimal A_∞ -module over A which is A_∞ -quasi-isomorphic to M .*

Let A be a strictly unital A_∞ -algebra over R . Let $\text{Mod}_\infty(A)$ be the category of strictly unital A_∞ -modules over A with strictly unital A_∞ -morphisms as morphisms. The *derived category* $\mathcal{D}(A)$ is the category obtained from $\text{Mod}_\infty(A)$ by formally inverting all A_∞ -quasi-isomorphisms. The category $\mathcal{D}(A)$ is a triangulated category whose suspension functor is the shift functor Σ . It has arbitrary (set-indexed) direct sums.

Let A be a dg algebra over R . There are three classes of modules we can consider. First we can view A as a strictly unital A_∞ -algebra over R with vanishing m_n for $n \geq 3$ and consider strictly unital A_∞ -modules over A . Secondly, we can consider unital dg A -modules (see [14, Section 2.1.1]), which are exactly the strictly unital A_∞ -modules M over A with $m_n^M = 0$ for $n \geq 3$. Thirdly, we can consider dg modules over A which is considered a dg k -algebra. It is easy to check that the second and the third classes coincide. By [14, Lemme 4.1.3.8], the derived category of dg A -modules is canonically equivalent to the derived category of strictly unital A_∞ -modules over A . We will identify these two derived categories.

Theorem 3.3. ([14, Théorème 4.1.2.4]) *Let $f : A \rightarrow B$ be a strictly unital A_∞ -quasi-isomorphism of strictly unital A_∞ -algebras over R . Then there is a triangle equivalence $f^* : \mathcal{D}(B) \rightarrow \mathcal{D}(A)$ which takes B_B to an A_∞ -module isomorphic to A_A in $\mathcal{D}(A)$.*

Let A be a strictly unital A_∞ -algebra over R . Let e be a strict idempotent of A . For a strictly unital A_∞ -module M over A let $Me = \{me := m_2^M(m \otimes e) | m \in M\}$. Then applying the identity (3.3) for $n = 2$ to $m \otimes e$, we get

$$m_1(me) = (-1)^{|m|} mm_1(e) + m_1(m)e = m_1(m)e \in Me.$$

So Me is a subcomplex of M . We will need the following result.

Lemma 3.4. *Let e be a strict idempotent of A . For a strictly unital A_∞ -module M over A and an integer p , there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{D}(A)}(eA, \Sigma^p M) \cong H^p(Me). \quad (3.5)$$

Proof. This can be obtained as a consequence of a suitable version of Yoneda's lemma. We do not find a reference in the literature, so we give a direct proof here. It is enough to prove for the case $p = 0$. By [14, Théorème 4.1.3.1], $\mathrm{Hom}_{\mathcal{D}(A)}(eA, M)$ is the same as the space of strictly unital A_∞ -morphisms from eA to M modulo those homotopic to 0. We show in four steps that this space is canonically isomorphic to $H^0(Me)$. Here two strictly unital A_∞ -morphisms $f, g : M \rightarrow M'$ of strictly unital A_∞ -modules are homotopic if there is a strictly unital homotopy h between f and g , that is, a family of homogeneous K -linear maps $h_n : M \otimes_R A^{\otimes_R(n-1)} \rightarrow M'$ ($n \geq 1$) of degree $-n$ such that $h_n(\mathrm{id}_M \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \eta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) = 0$ for all $n \geq 2$ and that the following identity holds for all $n \geq 1$

$$f_n - g_n = \sum_{s+t=n} (-1)^t m_{1+t}(h_s \otimes \mathrm{id}^{\otimes t}) + \sum_{i+j+l=n} (-1)^{ij+l} h_{i+1+l}(\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes l}). \quad (3.6)$$

Step 1: Let $m \in Z^0(Me)$. For $n \geq 1$, define a homogeneous R -linear map of degree $1 - n$

$$\begin{aligned} f_n : eA \otimes_R A^{\otimes_R(n-1)} &\longrightarrow M \\ a_1 \otimes a_2 \otimes \cdots \otimes a_n &\longmapsto (-1)^{n+1} m_{n+1}(m \otimes a_1 \otimes \cdots \otimes a_n). \end{aligned}$$

Then $f = (f_n)_{n \geq 1}$ is a strictly unital A_∞ -morphism from eA to M .

The identity $f_n(\mathrm{id}_{eA} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \eta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) = 0$ ($n \geq 2$) is clear since A is strictly unital. We need to check the identity (3.4) for all $n \geq 1$ applied to $a_1 \otimes a_2 \otimes \cdots \otimes a_n$, where $a_1 \in eA$ and $a_2, \dots, a_n \in A$ are homogenous. We have (note that when flipping tensors we

have the Koszul sign: $(\varphi \otimes \psi)(u \otimes v) = (-1)^{|\psi| \cdot |u|}(\varphi(u) \otimes \psi(v))$

$$\begin{aligned}
\text{LHS} &= \sum_{i+j+l=n} (-1)^{ij+l} f_{i+1+l}((-1)^{(|a_1|+\dots+|a_i|)(2-j)} a_1 \otimes \dots \otimes a_i \\
&\quad \otimes m_j(a_{i+1} \otimes \dots \otimes a_{i+j}) \otimes a_{i+j+1} \otimes \dots \otimes a_n) \\
&= \sum_{i+j+l=n} (-1)^{ij+l} (-1)^{i+2+l} m_{i+2+l}((-1)^{(|a_1|+\dots+|a_i|)(2-j)} m \otimes a_1 \otimes \dots \otimes a_i \\
&\quad \otimes m_j(a_{i+1} \otimes \dots \otimes a_{i+j}) \otimes a_{i+j+1} \otimes \dots \otimes a_n) \\
&= \sum_{i+j+l=n+1, i \geq 1} (-1)^{ij+l+n} m_{i+1+l}(\text{id}^{\otimes i} \otimes m_j \otimes \text{id}^{\otimes l})(m \otimes a_1 \otimes \dots \otimes a_n) \\
\text{RHS} &= \sum_{s+t=n} m_{1+t}(f_s(a_1 \otimes \dots \otimes a_s) \otimes a_{s+1} \otimes \dots \otimes a_n) \\
&= \sum_{s+t=n} (-1)^{s+1} m_{1+t}(m_{s+1}(m \otimes a_1 \otimes \dots \otimes a_s) \otimes a_{s+1} \otimes \dots \otimes a_n) \\
&= \sum_{s+t=n+1, s \geq 2} (-1)^{n+1+t} m_{1+t}(m_s \otimes \text{id}^{\otimes t})(m \otimes a_1 \otimes \dots \otimes a_n).
\end{aligned}$$

By (3.3), we have

$$\begin{aligned}
\text{LHS} - \text{RHS} &= -m_{1+n}(m_1 \otimes \text{id}^{\otimes n})(m \otimes a_1 \otimes \dots \otimes a_n) \\
&= -m_{1+n}(m_1(m) \otimes a_1 \otimes \dots \otimes a_n) \\
&= 0.
\end{aligned}$$

Step 2: Let $m \in Z^0(Me)$. In Step 1, we associate to m a strictly unital A_∞ -morphism f from eA to M . We claim that f is homotopic to 0 if and only if m belongs to $B^0(Me)$.

We first show that for any $m \in Me$ we have $m = me$. Suppose $m = m'e$. Applying (3.3) for $n = 3$ to $m' \otimes e \otimes e$ we get

$$m = m'e = m'(ee) = (m'e)e = me.$$

Now we prove the ‘only if’ part. Assume that f is homotopic to 0. Then there exists a homogeneous R -linear map $h_1 : eA \rightarrow M$ of degree -1 such that $f_1 = m_1 h_1 + h_1 m_1$. So

$$m = me = f_1(e) = m_1 h_1(e) + h_1 m_1(e) = m_1 h_1(e) \in B^0(Me).$$

Next we prove the ‘if’ part. Assume that $m = m_1(m')$, where $m' \in M^{-1}$. For $n \geq 1$, define a homogeneous R -linear map of degree $-n$

$$\begin{aligned}
h_n : eA \otimes_R A^{\otimes_R(n-1)} &\longrightarrow M \\
a_1 \otimes a_2 \otimes \dots \otimes a_n &\longmapsto m_{n+1}(m' \otimes a_1 \otimes \dots \otimes a_n).
\end{aligned}$$

Then $h = (h_n)_{n \geq 1}$ is a strictly unital homotopy between f and 0. The identity $h_n(\text{id}_{eA} \otimes \text{id} \otimes \dots \otimes \text{id} \otimes \eta \otimes \text{id} \otimes \dots \otimes \text{id}) = 0$ ($n \geq 2$) is clear since A is strictly unital. We need to check the

identify (3.6) for all $n \geq 1$ applied to $a_1 \otimes a_2 \otimes \cdots \otimes a_n$, where $a_1 \in eA$ and $a_2, \dots, a_n \in A$ are homogeneous. We have

$$\begin{aligned}
\text{RHS} &= \sum_{s+t=n} (-1)^t m_{1+t}(h_s(a_1 \otimes \cdots \otimes a_s) \otimes a_{s+1} \otimes \cdots \otimes a_n) \\
&\quad + \sum_{i+j+l=n} (-1)^{ij+l} h_{i+1+l}((-1)^{(|a_1|+\dots+|a_i|)(2-j)} a_1 \otimes \cdots \otimes a_i \\
&\quad \quad \otimes m_j(a_{i+1} \otimes \cdots \otimes a_{i+j}) \otimes a_{i+j+1} \otimes \cdots \otimes a_n) \\
&= \sum_{s+t=n} (-1)^t m_{1+t}(m_{s+1}(m' \otimes a_1 \otimes \cdots \otimes a_s) \otimes a_{s+1} \otimes \cdots \otimes a_n) \\
&\quad + \sum_{i+j+l=n} (-1)^{(i+1)j+l} m_{i+2+l}((-1)^{(|m'|+|a_1|+\dots+|a_i|)(2-j)} m' \otimes a_1 \otimes \cdots \otimes a_i \\
&\quad \quad \otimes m_j(a_{i+1} \otimes \cdots \otimes a_{i+j}) \otimes a_{i+j+1} \otimes \cdots \otimes a_n) \\
&= \sum_{s+t=n+1, s \geq 2} (-1)^t m_{1+t}(m_s \otimes \text{id}^{\otimes t})(m' \otimes a_1 \otimes \cdots \otimes a_n) \\
&\quad + \sum_{i+j+l=n+1, i \geq 1} (-1)^{ij+l} m_{i+1+l}(\text{id}^{\otimes i} \otimes m_j \otimes \text{id}^{\otimes l})(m' \otimes a_1 \otimes \cdots \otimes a_n) \\
&\stackrel{(3.3)}{=} -(-1)^n m_{1+n}(m_1 \otimes \text{id}^{\otimes n})(m' \otimes a_1 \otimes \cdots \otimes a_n) \\
&= (-1)^{n+1} m_{n+1}(m \otimes a_1 \otimes \cdots \otimes a_n) \\
&= f_n(a_1 \otimes \cdots \otimes a_n) = \text{LHS}.
\end{aligned}$$

Step 3: Let f be a strictly unital A_∞ -morphism from eA to M . We first show that $f_1(e) \in Z^0(Me)$. The identity (3.4) for $n = 1$ applied to e yields $m_1(f_1(e)) = f_1(m_1(e)) = 0$, so $f_1(e) \in Z^0(M)$. The same identity for $n = 2$ applied to $e \otimes e$ yields $f_1(e) = f_1(e)e + m_1 f_2(e \otimes e) = f_1(e)e \in Me$. The last equality holds because $f_2(\text{id} \otimes \eta) = 0$ and $e \in \text{im}(\eta)$.

We claim that f is homotopic to the strictly unital A_∞ -morphism from eA to M associated to $f_1(e)$ as in Step 1. It is enough to show that if $f_1(e) = 0$, then f is homotopic to 0.

For $n \geq 1$, define a homogeneous R -linear map of degree $-n$

$$\begin{aligned}
h_n : eA \otimes_R A^{\otimes_R(n-1)} &\longrightarrow M \\
a_1 \otimes a_2 \otimes \cdots \otimes a_n &\longmapsto (-1)^{n+1} f_{n+1}(e \otimes a_1 \otimes \cdots \otimes a_n).
\end{aligned}$$

Then $h = (h_n)_{n \geq 1}$ is a strictly unital homotopy between f and 0. The identity $h_n(\text{id}_{eA} \otimes \text{id} \otimes \cdots \otimes \text{id} \otimes \eta \otimes \text{id} \otimes \cdots \otimes \text{id}) = 0$ ($n \geq 2$) is clear since f is strictly unital. We need to check the identity (3.6) for all $n \geq 1$ applied to $a_1 \otimes a_2 \otimes \cdots \otimes a_n$, where $a_1 \in eA$ and $a_2, \dots, a_n \in A$ are

homogeneous. We have

$$\begin{aligned}
\text{RHS} &= \sum_{s+t=n} (-1)^t m_{1+t}(h_s(a_1 \otimes \cdots \otimes a_s) \otimes a_{s+1} \otimes \cdots \otimes a_n) \\
&\quad + \sum_{i+j+l=n} (-1)^{ij+l} h_{i+1+l}((-1)^{(|a_1|+\dots+|a_i|)(2-j)} a_1 \otimes \cdots \otimes a_i \\
&\quad \quad \otimes m_j(a_{i+1} \otimes \cdots \otimes a_{i+j}) \otimes a_{i+j+1} \otimes \cdots \otimes a_n) \\
&= \sum_{s+t=n} (-1)^t m_{1+t}((-1)^{s+1} f_{s+1}(e \otimes a_1 \otimes \cdots \otimes a_s) \otimes a_{s+1} \otimes \cdots \otimes a_n) \\
&\quad + \sum_{i+j+l=n} (-1)^{ij+l} (-1)^{i+2+l} f_{i+2+l}((-1)^{(|a_1|+\dots+|a_i|)(2-j)} e \otimes a_1 \otimes \cdots \otimes a_i \\
&\quad \quad \otimes m_j(a_{i+1} \otimes \cdots \otimes a_{i+j}) \otimes a_{i+j+1} \otimes \cdots \otimes a_n) \\
&= \sum_{s+t=n+1, s \geq 2} (-1)^{n+1} m_{1+t}(f_s \otimes \text{id}^{\otimes t})(e \otimes a_1 \otimes \cdots \otimes a_n) \\
&\quad + \sum_{i+j+l=n} (-1)^{(i+1)j+l+n} f_{i+2+l}(\text{id}^{\otimes(i+1)} \otimes m_j \otimes \text{id}^{\otimes l})(e \otimes a_1 \otimes \cdots \otimes a_n) \\
&\stackrel{f_1(e)=0}{=} \sum_{s+t=n+1} (-1)^{n+1} m_{1+t}(f_s \otimes \text{id}^{\otimes t})(e \otimes a_1 \otimes \cdots \otimes a_n) \\
&\quad + \sum_{i+j+l=n+1, i \geq 1} (-1)^{ij+l+n} f_{i+1+l}(\text{id}^{\otimes i} \otimes m_j \otimes \text{id}^{\otimes l})(e \otimes a_1 \otimes \cdots \otimes a_n) \\
&\stackrel{(3.4)}{=} -(-1)^n \sum_{j+l=n+1} (-1)^l f_{1+l}(m_j \otimes \text{id}^{\otimes l})(e \otimes a_1 \otimes \cdots \otimes a_n) \\
&= f_n(m_2 \otimes \text{id}^{\otimes n})(e \otimes a_1 \otimes \cdots \otimes a_n) \\
&= f_n(a_1 \otimes \cdots \otimes a_n) = \text{LHS}.
\end{aligned}$$

Step 4: By Step 3, $\text{Hom}_{\mathcal{D}(A)}(eA, M)$ is canonically isomorphic to the space of strictly unital A_∞ -morphisms from eA to M associated to $m \in Z^0(Me)$ modulo those homotopic to 0, and hence to $H^0(Me) = Z^0(Me)/B^0(Me)$ by Step 2. \checkmark

Denote by $\text{per}(A)$ the thick subcategory of $\mathcal{D}(A)$ generated by A_A , and denote by $\mathcal{D}_{fd}(A)$ the full subcategory of $\mathcal{D}(A)$ consisting of those A_∞ -modules whose total cohomology is finite-dimensional.

Lemma 3.5. (a) *Let M be an object of $\mathcal{D}(A)$. Then M is compact if and only if it belongs to $\text{per}(A)$, and M belongs to $\mathcal{D}_{fd}(A)$ if and only if $\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}(A)}(N, \Sigma^i M)$ is finite-dimensional for any $N \in \text{per}(A)$.*

(b) *Let A and B be two strictly unital A_∞ -algebras over R . A triangle equivalence $\mathcal{D}(A) \rightarrow \mathcal{D}(B)$ restricts to triangle equivalences $\text{per}(A) \rightarrow \text{per}(B)$ and $\mathcal{D}_{fd}(A) \rightarrow \mathcal{D}_{fd}(B)$.*

Proof. Let A' be a dg model of A as in Theorem 3.1. Then by Theorem 3.3 there is a triangle equivalence $\mathcal{D}(A') \rightarrow \mathcal{D}(A)$ which takes $A'_{A'}$ to A_A . By [10, Corollary 3.7] (also [8, Remark

5.3 (a)]), an object of $\mathcal{D}(A')$ is compact if and only if it belongs to $\text{per}(A')$. The first assertion of (a) follows immediately. The second assertion of (a) follows from Lemma 3.4 by dévissage. (b) is a direct consequence of (a) since $\text{per}(A)$ and $\mathcal{D}_{fd}(A)$ admit intrinsic descriptions inside $\mathcal{D}(A)$. \checkmark

We have the following Morita's theorems for derived categories and for algebraic triangulated categories. They can be considered as special cases of [14, Théorème 7.6.0.4]) and [14, Théorème 7.6.0.6], respectively.

Theorem 3.6. *Let A be a strictly unital A_∞ -algebra over R . Let $\{X_1, \dots, X_r\}$ be a set of compact generators of $\mathcal{D}(A)$, i.e. $\text{per}(A) = \text{thick}(X_1, \dots, X_r)$. Let K be the direct product of r copies of k . Then there is a strictly unital minimal A_∞ -algebra B over K such that as a graded algebra*

$$B = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}(A)} \left(\bigoplus_{i=1}^r X_i, \Sigma^p \bigoplus_{i=1}^r X_i \right),$$

and there is a triangle equivalence

$$\mathcal{D}(B) \longrightarrow \mathcal{D}(A)$$

taking $e_i B$ ($1 \leq i \leq r$) to X_i .

Proof. By [14, Théorème 7.6.0.4], there is a strictly unital minimal A_∞ -category \mathcal{B} whose objects are X_1, \dots, X_r , and whose morphism spaces are $\text{Hom}_{\mathcal{B}}(X_i, X_j) = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}(A)}(X_i, \Sigma^p X_j)$ for $1 \leq i, j \leq r$, and a triangle equivalence

$$\mathcal{D}(\mathcal{B}) \longrightarrow \mathcal{D}(A),$$

which takes $\widehat{X_i}$ to X_i for $1 \leq i \leq r$. We identify K with $k\{\text{id}_{X_1}\} \times \dots \times k\{\text{id}_{X_r}\}$. Let B be the graded K -bimodule $\bigoplus_{i,j=1}^r \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}(A)}(X_i, \Sigma^p X_j)$. Then the multiplications on \mathcal{B} induce multiplications on B such that B becomes a minimal strictly unital A_∞ -algebra over K . The assignment $M \mapsto \bigoplus_{i=1}^r M(X_i)$ extends to an isomorphism $\text{Mod}_\infty(\mathcal{B}) \rightarrow \text{Mod}_\infty(B)$, which induces a triangle isomorphism $\mathcal{D}(\mathcal{B}) \rightarrow \mathcal{D}(B)$. So we have the desired triangle equivalence. \checkmark

Theorem 3.7. *Let \mathcal{C} be an idempotent complete algebraic triangulated category. Assume that \mathcal{C} is generated by a set of objects $\{X_1, \dots, X_r\}$, i.e. $\mathcal{C} = \text{thick}(X_1, \dots, X_r)$. Let K be the direct product of r copies of k . Then there is a strictly unital minimal A_∞ -algebra B over K such that as a graded algebra*

$$B = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{C}} \left(\bigoplus_{i=1}^r X_i, \Sigma^p \bigoplus_{i=1}^r X_i \right),$$

and there is a triangle equivalence

$$\text{per}(B) \longrightarrow \mathcal{C}$$

taking $e_i B$ ($1 \leq i \leq r$) to X_i .

Proof. This is a consequence of [14, Théorème 7.6.0.6]. The proof is similar to that of the preceding result. \checkmark

4. SIMPLE-MINDED COLLECTIONS AND MINIMAL POSITIVE A_∞ -ALGEBRAS

In this section we recall the definition of simple-minded collections and study strictly unital minimal positive A_∞ -algebras, which are closely related to simple-minded collections.

4.1. Simple-minded collections. Let \mathcal{C} be a triangulated category with suspension functor Σ . A collection $\{X_1, \dots, X_r\}$ of objects of \mathcal{C} is *simple-minded* if

- $\text{Hom}_{\mathcal{C}}(X_i, \Sigma^p X_j) = 0$, $\forall p < 0$ and $1 \leq i, j \leq r$,
- $\text{Hom}_{\mathcal{C}}(X_i, X_j) = 0$ if $1 \leq i \neq j \leq r$ and $\text{End}_{\mathcal{C}}(X_i)$ is a division k -algebra for all $1 \leq i \leq r$,
- $\mathcal{C} = \text{thick}(X_1, \dots, X_r)$.

Two simple-minded collections $\{X_1, \dots, X_r\}$ and $\{X'_1, \dots, X'_r\}$ are said to be *isomorphic* if up to reordering we have $X_i \cong X'_i$ for all $1 \leq i \leq r$. A simple-minded collection $\{X_1, \dots, X_r\}$ is said to be *elementary* if $\text{End}_{\mathcal{C}}(X_i) \cong k$ for all $1 \leq i \leq r$. If the field k is algebraically closed, then every simple-minded collection in \mathcal{C} is elementary. In general \mathcal{C} may contain non-elementary simple-minded collections and it is not known whether any two simple-minded collections in \mathcal{C} have the same set of endomorphism algebras. It is the case when the two simple-minded collections are related by a mutation ([13, Remark 7.7]).

Let A be a non-positive dg k -algebra with $H^0(A)$ being finite-dimensional over k . Then a complete set of pairwise non-isomorphic simple $H^0(A)$ -modules, viewed as dg A -modules via the homomorphism $A \rightarrow H^0(A)$, form a simple-minded collection in $\mathcal{D}_{fd}(A)$, see for example [5, Theorem A.1 (c)].

4.2. Strictly unital minimal positive A_∞ -algebras. Fix $r \in \mathbb{N}$. Let K be the direct product of r copies of k . Let e_1, \dots, e_r be the standard basis of K over k .

Let A be a strictly unital minimal A_∞ -algebra over K . We say that A is *positive* if

- $A^p = 0$ for all $p < 0$,
- $A^0 = K$ and the unit is the embedding $K = A^0 \hookrightarrow A$.

It is clear that e_1, \dots, e_r are strict idempotents of A . So $e_1 A, \dots, e_r A$ are submodules of A . Moreover, as an A_∞ -module $A = \bigoplus_{i=1}^r e_i A$.

Lemma 4.1. (a) *Let A be a strictly unital minimal positive A_∞ -algebra over K . Then $\{e_1 A, \dots, e_r A\}$ is an elementary simple-minded collection in $\text{per}(A)$.*
 (b) *Let \mathcal{C} be an idempotent complete algebraic triangulated category and let $\{X_1, \dots, X_r\}$ be an elementary simple-minded collection in \mathcal{C} . Then there is a strictly unital minimal positive A_∞ -algebra A over K together with a triangle equivalence $\mathcal{C} \rightarrow \text{per}(A)$ which takes X_i ($1 \leq i \leq r$) to $e_i A$.*

Proof. (a) Put $P_i = e_i A$ ($1 \leq i \leq r$). Then by Lemma 3.4 we have

$$\begin{aligned} \cdot \text{Hom}(P_i, \Sigma^p P_j) &= H^p(e_j A e_i) = 0 \text{ for } p < 0, \\ \cdot \text{Hom}(P_i, P_j) &= H^0(e_j A e_i) = \begin{cases} k & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Moreover, P_1, \dots, P_r generates $\text{per}(A)$ since $A = P_1 \oplus \dots \oplus P_r$. Therefore P_1, \dots, P_r is an elementary simple-minded collection in $\text{per}(A)$.

(b) By Theorem 3.7, there is a strictly unital minimal A_∞ -algebra A over K such that as a graded algebra

$$A = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{C}} \left(\bigoplus_{i=1}^r X_i, \Sigma^p \bigoplus_{i=1}^r X_i \right),$$

and there is a triangle equivalence

$$\mathcal{C} \longrightarrow \text{per}(A)$$

taking X_i to $e_i A$, $1 \leq i \leq r$. That A is positive follows from the assumption that $\{X_1, \dots, X_r\}$ is an elementary simple-minded collection. \checkmark

Let A be a strictly unital minimal positive A_∞ -algebra over K . The projection $\varepsilon : A \rightarrow A^0 = K$ makes A an augmented A_∞ -algebra over K . We view K as an A_∞ -module over A via ε and denote it by S . For $1 \leq i \leq r$, we have a 1-dimensional A_∞ -module $S_i = e_i A / e_i \ker(\varepsilon)$. Then $S = S_1 \oplus \dots \oplus S_r$. We call S_1, \dots, S_r the *simple modules* over A . Let M be a strictly unital A_∞ -module over A which is concentrated in degree 0. Then for all $m \in M$ we have $m_n^M(m \otimes a_1 \otimes \dots \otimes a_{n-1}) = 0$ if a_1, \dots, a_{n-1} are homogenous and one of them belongs to $\ker(\varepsilon)$. Indeed, we may assume that $m \in M^0$. If at least one of a_1, \dots, a_{n-1} belongs to A^0 and at least one of them belongs to $\ker(\varepsilon)$, then $m_n^M(m \otimes a_1 \otimes \dots \otimes a_{n-1}) = 0$ because M is strictly unital. If all a_1, \dots, a_{n-1} belong to $\ker(\varepsilon)$, then $m_n^M(m \otimes a_1 \otimes \dots \otimes a_{n-1})$ is homogeneous of degree different from 0 and has to be zero. So the A_∞ -module structure on M factors through ε , so M is the direct sum of copies of simple modules.

Lemma 4.2. *Let M be a strictly unital A_∞ -module over A . Fix $1 \leq i \leq r$. If for $1 \leq j \leq r$ and $p \in \mathbb{Z}$ we have*

$$\text{Hom}_{\mathcal{D}(A)}(e_j A, \Sigma^p M) = \begin{cases} k & \text{if } j = i \text{ and } p = 0, \\ 0 & \text{otherwise,} \end{cases}$$

then M is A_∞ -quasi-isomorphic to S_i .

Proof. By Lemma 3.4, we have

$$H^p(M e_j) = \begin{cases} k & \text{if } j = i \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

So by Proposition 3.2, M is A_∞ -quasi-isomorphic to a strictly unital A_∞ -module M' , which is 1-dimensional and concentrated in degree 0. Moreover $M' e_i = k$ and $M' e_j = 0$ for $j \neq i$. So M' is isomorphic to S_i . \checkmark

The first statement of the following lemma is also proved in [11] (phrased in terms of dg algebras and dg modules).

Lemma 4.3. *The A_∞ -module S is a silting object in $\mathcal{D}_{fd}(A)$. Moreover, if A is Koszul as a graded algebra (see for example [3] for a definition), then S is a tilting object in $\mathcal{D}_{fd}(A)$.*

Proof. We first show that $\mathrm{Hom}_{\mathcal{D}(A)}(S, \Sigma^m S) = 0$ for $m > 0$. Take the enveloping algebra U of A as in Theorem 3.1. Then U is an augmented dg algebra over K and there is a strictly unital A_∞ -quasi-isomorphism $A \rightarrow U$ of augmented A_∞ -algebras over K . So S can be viewed as a dg U -module and the A_∞ -structure on S over A factors through the A_∞ -quasi-isomorphism $A \rightarrow U$. By Theorem 3.3, there is a triangle equivalence $\mathcal{D}(U) \rightarrow \mathcal{D}(A)$ which sends S to S . So we only need to show that $\mathrm{Hom}_{\mathcal{D}(U)}(S, \Sigma^m S) = 0$ for $m > 0$. We view S as a graded module over the graded algebra $H^*(U)$, which is identified with A viewed as a graded algebra. S admits a projective resolution over the graded algebra A

$$\dots \rightarrow P^m \rightarrow P^{m+1} \rightarrow \dots \rightarrow P^{-1} \rightarrow P^0$$

such that $P^m \in \mathrm{Add}(A\langle m' \rangle \mid m' \leq m)$, where $\langle 1 \rangle$ denotes the degree shift. According to [8, Theorem 3.1 (c)], there is a dg module P over U which is quasi-isomorphic to S and which admits a filtration

$$0 = F_{-1} \subset F_0 \subset \dots \subset F_p \subset F_{p+1} \subset \dots \subset P, \quad p \in \mathbb{N}$$

such that

- (F1) P is the union of the F_p , $p \in \mathbb{N}$.
- (F2) $\forall p \in \mathbb{N}$, the inclusion morphism $F_{p-1} \subset F_p$ splits in the category $\mathrm{Grmod} U$ of graded modules over U , which is considered as a graded algebra by forgetting the differential.
- (F3) $\forall p \in \mathbb{N}$, $F_p/F_{p-1} \in \mathrm{Add}(\Sigma^m A \mid m \leq 0)$.

By (F1) and (F2), we have an isomorphism $P \cong \bigoplus_{p \geq 0} F_p/F_{p-1}$ in $\mathrm{Grmod} U$. So as a graded vector space $\mathcal{H}om_U(P, S) = \prod_{p \geq 0} \mathcal{H}om_U(F_p/F_{p-1}, S)$, which is concentrated in non-positive degrees by (F3), since $\mathcal{H}om_U(\Sigma^m U, S) = \Sigma^{-m} S$. As a consequence, we obtain that for $m > 0$

$$\mathrm{Hom}_{\mathcal{D}(U)}(S, \Sigma^m S) = \mathrm{Hom}_{\mathcal{D}(U)}(P, \Sigma^m S) = H^m \mathcal{H}om_U(P, S) = 0.$$

Next we show that $\mathcal{D}_{fd}(A) = \mathrm{thick}(S)$. By Proposition 3.2, it suffices to prove that if a strictly unital minimal A_∞ -module M over A satisfies that $\dim(M) := \bigoplus_{m \in \mathbb{Z}} M^m$ is finite-dimensional, then $M \in \mathrm{thick}(S)$. Up to shift we may assume that $M^m = 0$ for all $m < 0$ and $M^0 \neq 0$. Define $M^{>0} = \bigoplus_{m > 0} M^m$. Then $M^{>0}$ is a submodule of M . Let $\iota : M^{>0} \rightarrow M$ be the embedding and form a triangle in $\mathcal{D}(A)$

$$M^{>0} \xrightarrow{\iota} M \longrightarrow \bar{M} \longrightarrow \Sigma M^{>0}.$$

Here \bar{M} is assumed to be minimal. Looking at the long exact sequence of cohomologies, we see that \bar{M} is concentrated in degree 0, and hence is a finite direct sum of copies of S_1, \dots, S_r . Now by induction on $\dim(M)$ we finish the proof.

Finally, assume that A is Koszul. Then the above resolution of S can be chosen such that $P^m \in \mathrm{Add}(A\langle m \rangle)$. Consequently, $F_p/F_{p-1} \in \mathrm{Add}(U)$ and $\mathcal{H}om_U(P, S)$ is concentrated in degree 0. It follows that $\mathrm{Hom}_{\mathcal{D}(U)}(S, \Sigma^m S) = 0$ for $m \neq 0$. \checkmark

5. CONSTRUCTING SILTING OBJECTS FROM SIMPLES-MINDED COLLECTIONS

In this section we will use Koszul duality to construct a silting object in the perfect derived category of a finite-dimensional non-positive dg algebra from a given simple-minded collection in the finite-dimensional derived category.

Let A be a non-positive dg k -algebra whose total cohomology is finite-dimensional over k . In particular, $H^0(A)$ is a finite-dimensional algebra. Let S_1, \dots, S_r be a complete set of pairwise non-isomorphic simple $H^0(A)$ -modules and view them as dg A -modules via the homomorphism $A \rightarrow H^0(A)$. Recall that $\{S_1, \dots, S_r\}$ is a simple-minded collection in $\mathcal{D}_{fd}(A)$. We assume further that $\text{End}_{H^0(A)}(S_i) \cong k$ for all $1 \leq i \leq r$. Then $\{S_1, \dots, S_r\}$ is an elementary simple-minded collection in $\mathcal{D}_{fd}(A)$.

Since $\text{End}_{\mathcal{D}(A)}(A) = H^0(A)$, it follows that there are indecomposable objects $P_1, \dots, P_r \in \text{per}(A)$ and positive integers a_1, \dots, a_r such that $A \cong \bigoplus_{i=1}^r P_i^{\oplus a_i}$ in $\mathcal{D}(A)$ and that

$$\text{Hom}_{\mathcal{D}(A)}(P_i, \Sigma^p S_j) = \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The collection $\{S_1, \dots, S_r\}$ is determined by this property. Namely, let $1 \leq j \leq r$ and $M \in \mathcal{D}(A)$ be such that

$$\text{Hom}_{\mathcal{D}(A)}(P_i, \Sigma^p M) = \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise,} \end{cases}$$

then $M \cong S_j$ in $\mathcal{D}(A)$. Indeed, this property implies that $H^p(M) \cong \text{Hom}_{\mathcal{D}(A)}(A, \Sigma^p M)$ is trivial unless $p = 0$, so M is isomorphic in $\mathcal{D}(A)$ to $H^0(M)$, which is a dg A -module via the homomorphism $A \rightarrow H^0(A)$. Moreover, we have

$$\text{Hom}_{H^0(A)}(H^0(P_i), H^0(M)) = \begin{cases} k & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $H^0(M) \cong S_j$ in $\text{mod } H^0(A)$. Therefore $M \cong S_j$ in $\mathcal{D}(A)$.

Let $I_i = \nu(P_i)$ for $1 \leq i \leq r$. Then $D(AA) = \bigoplus_{i=1}^r I_i^{\oplus a_i}$ and by the Auslander–Reiten formula we have

$$\text{Hom}_{\mathcal{D}(A)}(S_i, \Sigma^p I_j) = \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let K be the direct sum of r copies of k and let e_1, \dots, e_r be the standard basis of K over k . By Lemma 4.1 (b), there is a strictly unital minimal positive A_∞ -algebra \mathcal{S} over K such that as a graded algebra

$$\mathcal{S} = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}_{fd}(A)}\left(\bigoplus_{i=1}^r S_i, \Sigma^p \bigoplus_{j=1}^r S_j\right)$$

and there is a triangle equivalence

$$\Phi : \mathcal{D}_{fd}(A) \longrightarrow \text{per}(\mathcal{S})$$

taking S_i ($1 \leq i \leq r$) to $e_i \mathcal{S}$. Therefore we have

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{S})}(e_i \mathcal{S}, \Sigma^p \Phi(I_j)) = \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 4.2, $\Phi(I_1), \dots, \Phi(I_r)$ are, up to A_∞ -quasi-isomorphism, precisely the simple modules over \mathcal{S} . In other words, the equivalence Φ restricts to a triangle equivalence

$$\Phi| : \mathrm{thick}_{\mathcal{D}(A)}(D(AA)) = \mathrm{thick}_{\mathcal{D}(A)}(I_1, \dots, I_r) \longrightarrow \mathrm{thick}_{\mathcal{D}(\mathcal{S})}(\Phi(I_1), \dots, \Phi(I_r)) = \mathcal{D}_{fd}(\mathcal{S}),$$

where the last equality follows from Lemma 4.3. It follows that $\mathcal{D}_{fd}(\mathcal{S}) \subseteq \mathrm{per}(\mathcal{S})$.

Let $\{X_1, \dots, X_r\} \subseteq \mathcal{D}_{fd}(A)$ be an elementary simple-minded collection. Let $Y_i = \Phi(X_i)$ for $1 \leq i \leq r$. Then $\{Y_1, \dots, Y_r\}$ is an elementary simple-minded collection in $\mathrm{per}(\mathcal{S})$. By Theorem 3.6, there is a strictly unital minimal positive A_∞ -algebra \mathcal{X} over K such that as a graded algebra

$$\mathcal{X} = \bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}_{fd}(A)}\left(\bigoplus_{i=1}^r Y_i, \Sigma^p \bigoplus_{i=1}^r Y_i\right)$$

and there is a triangle equivalence

$$\tilde{\Psi} : \mathcal{D}(\mathcal{S}) \longrightarrow \mathcal{D}(\mathcal{X})$$

taking Y_i ($1 \leq i \leq r$) to $e_i \mathcal{X}$. By Lemma 3.5, $\tilde{\Psi}$ restricts to triangle equivalences

$$\Psi : \mathrm{per}(\mathcal{S}) \longrightarrow \mathrm{per}(\mathcal{X}),$$

$$\Psi| : \mathcal{D}_{fd}(\mathcal{S}) \longrightarrow \mathcal{D}_{fd}(\mathcal{X}).$$

This implies that $\mathcal{D}_{fd}(\mathcal{X}) \subseteq \mathrm{per}(\mathcal{X})$. So we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{D}_{fd}(A) & \xrightarrow{\Phi} & \mathrm{per}(\mathcal{S}) & \xrightarrow{\Psi} & \mathrm{per}(\mathcal{X}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{thick}(D(AA)) & \xrightarrow{\Phi|} & \mathcal{D}_{fd}(\mathcal{S}) & \xrightarrow{\Psi|} & \mathcal{D}_{fd}(\mathcal{X}) \end{array}$$

Let R_1, \dots, R_r be the simple modules over \mathcal{X} , and let T_1, \dots, T_r be their images under a quasi-inverse of the equivalence $(\Psi \circ \Phi)|$. Put $T = \bigoplus_{i=1}^r T_i$.

Proposition 5.1. (a) T is a silting object of $\mathrm{thick}(D(AA))$.

(b) For $1 \leq i, j \leq r$, and $p \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathcal{D}_{fd}(A)}(X_j, \Sigma^p T_i) = \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(c) $\nu^{-1}T$ is a silting object of $\mathrm{per}(A)$.

(d) For $1 \leq i, j \leq r$, and $m \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathcal{D}_{fd}(A)}(\nu^{-1}T_i, \Sigma^p X_j) = \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (a) This is because $R_1 \oplus \dots \oplus R_r$ is a silting object of $\mathcal{D}_{fd}(\mathcal{X})$ (Lemma 4.3) and $(\Psi \circ \Phi)|$ is a triangle equivalence.

(b) By Lemma 3.4 we have

$$\mathrm{Hom}(e_j \mathcal{X}, \Sigma^p R_i) = \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The desired formula follows immediately because $\Psi \circ \Phi$ is a triangle equivalence.

(c) follows from (a) because $\nu : \mathrm{per}(A) \rightarrow \mathrm{thick}(D({}_A A))$ is a triangle equivalence.

(d) follows from (b) and the Auslander–Reiten formula. \checkmark

If A is a finite-dimensional elementary (ordinary) k -algebra, then this is [13, Lemmas 5.6, 5.7 and 5.8 and Proposition 5.9], up to the hypothesis that if $\mathcal{D}^b(\mathrm{mod} A) = \mathcal{D}_{fd}(A)$ has an elementary simple-minded collection, then all simple-minded collections in $\mathcal{D}^b(\mathrm{mod} A)$ are elementary. Compared with Rickard’s construction [18] used in [13], our new approach has the disadvantage that it may fail for non-elementary simple-minded collections, but it also has some advantages. By Lemma 4.3, we obtain a sufficient condition on $\nu^{-1}T$ being a tilting object.

Corollary 5.2. *If \mathcal{X} as a graded algebra is Koszul, then $\nu^{-1}T$ is a tilting object of $\mathrm{per}(A)$.*

By [20], if \mathcal{X} as a graded algebra is isomorphic to kQ , where Q is a graded quiver with all arrows in positive degrees, then $\mathcal{D}_{fd}(A)$ is triangle equivalent to $\mathrm{per}(kQ)$, where kQ is considered as a dg algebra with trivial differential. If all arrows of Q are in degree 1, then it is known that $\mathrm{per}(kQ)$ is triangle equivalent to the radical-square-zero algebra R associated to the opposite quiver Q^{op} , considered as an ungraded quiver, see for example [6, Theorem 2.5]. Consequently, A is derived equivalent to R .

Furthermore, the dg endomorphism algebra and the truncated dg endomorphism algebra $\tilde{\Gamma}$ of $\nu^{-1}T$ (see Section 2.3) are Koszul dual to the A_∞ -algebra \mathcal{X} . So they can be obtained, up to quasi-equivalence (in the sense of [8, Section 7]), as the dual bar construction of \mathcal{X} : the complete tensor algebra $B^\# \mathcal{S} = \widehat{T}_K(D(\mathcal{X}^{>0}[1]))$ of $D(\mathcal{X}^{>0}[1])$ over K (see [15, 7]), where $\mathcal{X}^{>0} = \bigoplus_{p>0} \mathcal{X}^p$. Namely, up to quasi-equivalence the following diagram is commutative

$$\begin{array}{ccc} \{X_1, \dots, X_r\} & \xrightarrow{\quad} & \nu^{-1}T \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow[\text{construction}]{\text{dual bar}} & \tilde{\Gamma}. \end{array}$$

In general the A_∞ -structure on \mathcal{X} is hard to compute. However, sometimes it is easy to obtain the A_∞ -structure on the truncated part $\mathcal{X}^{[0,2]}$, the A_∞ -algebra obtained from \mathcal{X} by modulo the elements of degree ≥ 3 . We have

$$\mathrm{End}_{\mathcal{D}(A)}(\nu^{-1}T) = H^0(\mathcal{E}nd_A(\nu^{-1}T)) = H^0(B^\# \mathcal{X}) = H^0(B^\# \mathcal{X}^{[0,2]}).$$

The quiver Q of $\mathrm{End}_{\mathcal{D}(A)}(\nu^{-1}T)$ is determined by the K -bimodule structure on \mathcal{X}^1 . Precisely, the set of vertices of Q is $\{1, \dots, r\}$, and the number of arrows from i to j is the dimension of

$e_i \mathcal{X}^1 e_j$ over k . The relations of $\text{End}_{\mathcal{D}(A)}(\nu^{-1}T)$ are ‘dual’ to the restrictions $m_n : (\mathcal{X}^1)^{\otimes_{K^n}} \rightarrow \mathcal{X}^2$ of the multiplications of \mathcal{X} .

Corollary 5.3. *If $\mathcal{X}^1 = 0$, then as an algebra $\text{End}_{\mathcal{D}(A)}(\nu^{-1}T)$ is isomorphic to K .*

The following example is taken from [2]. Let A be the algebra given by the quiver $1 \xrightleftharpoons[\beta]{\alpha} 2$ with relations $\alpha\beta = 0 = \beta\alpha$. Take $X_1 = P_1$ and $X_2 = \Sigma^{-1}S_1$. The \mathcal{X} as a graded algebra is the path algebra of the graded quiver

$$1 \xrightarrow{\gamma} 2 \circlearrowright \delta$$

where γ is of degree 1 and δ is of degree 2. Simply because of lack of morphisms to multiply with, the A_∞ -structure on $\mathcal{X}^{[0,2]}$ is trivial. The dual bar construction shows that $\text{End}_{\mathcal{D}(A)}(\nu^{-1}T)$ is the path algebra of the ungraded quiver $1 \longleftarrow 2$.

Now we state our main result, which is a consequence of Proposition 5.1.

Theorem 5.4. *For an elementary simple-minded collection $\{X_1, \dots, X_r\}$ of $\mathcal{D}_{fd}(A)$, there exists a unique (up to isomorphism) silting object $M = M_1 \oplus \dots \oplus M_r$ of $\text{per}(A)$ such that for $1 \leq i, j \leq r$ and $p \in \mathbb{Z}$*

$$\text{Hom}_{\mathcal{D}_{fd}(A)}(M_i, \Sigma^p X_j) = \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The existence of M follows from Proposition 5.1 (c)(d): take $M = \nu^{-1}T$. Let $N = N_1 \oplus \dots \oplus N_r$ be an object of $\text{per}(A)$ such that for $1 \leq i, j \leq r$ and $p \in \mathbb{Z}$

$$\text{Hom}_{\mathcal{D}_{fd}(A)}(N_i, \Sigma^p X_j) = \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then by the Auslander–Reiten formula we have

$$\text{Hom}_{\mathcal{D}_{fd}(A)}(\Sigma^p X_j, \nu N_i) = \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Applying the triangle equivalence $\Psi \circ \Phi$ we obtain

$$\text{Hom}_{\mathcal{D}_{fd}(A)}(\Sigma^p e_j \mathcal{X}, \Psi \circ \Phi \circ \nu(N_i)) = \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore by Lemma 4.2 we have $\Psi \circ \Phi \circ \nu(N_i) \cong R_i$. So $N_i \cong \nu^{-1} \circ (\Psi \circ \Phi)^{-1}(R_i) = M_i$, showing the uniqueness of M . ✓

As a consequence, we obtain the following theorem, generalising [13, Theorems 6.1 and 7.12].

Theorem 5.5. *Assume that k is algebraically closed. Then there are one-to-one correspondences which commute with mutations and which preserve partial orders between*

- (1) *equivalence classes of silting objects in $\text{per}(A)$,*

- (2) isomorphism classes of simple-minded collections in $\mathcal{D}_{fd}(A)$,
- (3) bounded t -structures of $\mathcal{D}_{fd}(A)$ with length heart,
- (4) bounded co- t -structures of $\mathbf{per}(A)$.

Proof. The proof is the same as that of [13, Theorems 6.1 and 7.12]. Here we only give the definition of some of the correspondences, which are compatible with each other.

From silting objects to simple-minded collections: Let M be a basic silting object in $\mathbf{per}(A)$. We may assume that M is cofibrant. Let $\tilde{\Gamma}$ be the truncated dg endomorphism algebra of M (see Section 2.3). Then $\tilde{\Gamma}$ is non-positive and has finite-dimensional total cohomology; moreover, there is a triangle equivalence $\mathcal{D}(\tilde{\Gamma}) \rightarrow \mathcal{D}(A)$ taking $\tilde{\Gamma}$ to M . The simple-minded collection $\{X_1, \dots, X_r\}$ corresponding to M is the image of a complete collection of pairwise non-isomorphic simple $H^0(\tilde{\Gamma})$ -modules (viewed as dg $\tilde{\Gamma}$ -modules) under this equivalence. It is the unique collection (up to isomorphism) in $\mathcal{D}_{fd}(A)$ satisfying for $1 \leq i, j \leq r$ and $p \in \mathbb{Z}$

$$\mathrm{Hom}(M_i, \Sigma^p X_j) = \begin{cases} k & \text{if } i = j \text{ and } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

From silting objects to t -structures: Let M be a silting object in $\mathbf{per}(A)$. The corresponding t -structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}_{fd}(A)$ is defined as

$$\begin{aligned} \mathcal{D}^{\leq 0} &= \{X \in \mathcal{D}_{fd}(A) \mid \mathrm{Hom}(M, \Sigma^p X) = 0 \text{ for } p > 0\}, \\ \mathcal{D}^{\geq 0} &= \{X \in \mathcal{D}_{fd}(A) \mid \mathrm{Hom}(M, \Sigma^p X) = 0 \text{ for } p < 0\}. \end{aligned}$$

This is the image of the standard t -structure ([5, Theorem A.1]) on $\mathcal{D}_{fd}(\tilde{\Gamma})$ under the triangle equivalence $\mathcal{D}_{fd}(\tilde{\Gamma}) \rightarrow \mathcal{D}_{fd}(A)$ (which is restricted from the triangle equivalence $\mathcal{D}(\tilde{\Gamma}) \rightarrow \mathcal{D}(A)$). The heart of this t -structure is equivalent to $\mathbf{mod} \mathbf{End}(M)$.

From silting objects to co- t -structures: Let M be a silting object in $\mathbf{per}(A)$. The corresponding co- t -structure $(\mathcal{P}_{\geq 0}, \mathcal{P}_{\leq 0})$ on $\mathbf{per}(A)$ is defined as (see [13, Section 3.4])

$$\begin{aligned} \mathcal{P}_{\geq 0} &= \text{the smallest full subcategory of } \mathbf{per}(A) \text{ which contains } \{\Sigma^p M \mid p \leq 0\} \\ &\quad \text{and which is closed under taking extensions and direct summands,} \\ \mathcal{P}_{\leq 0} &= \text{the smallest full subcategory of } \mathbf{per}(A) \text{ which contains } \{\Sigma^p M \mid p \geq 0\} \\ &\quad \text{and which is closed under taking extensions and direct summands.} \end{aligned}$$

From t -structures to simple-minded collections: Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t -structure on $\mathcal{D}_{fd}(A)$ with length heart. The corresponding simple-minded collection is a complete collection of pairwise non-isomorphic simple objects of the heart $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ (see [13, Section 3.3]).

From simple-minded collections to silting objects: Let $\{X_1, \dots, X_r\}$ be a simple-minded collection of $\mathcal{D}_{fd}(A)$. It is elementary since the base field k is algebraically closed. The corresponding silting object is the M as in Theorem 5.4.

From co- t -structures to silting objects: Let $(\mathcal{P}_{\geq 0}, \mathcal{P}_{\leq 0})$ be a bounded co- t -structure on $\mathbf{per}(A)$. The corresponding silting object M of $\mathbf{per}(A)$ is an additive generator of the co-heart, *i.e.* $\mathbf{add}(M) = \mathcal{P}_{\geq 0} \cap \mathcal{P}_{\leq 0}$ (see [13, Sections 3.1 and 3.4]). ✓

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